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The inverted weighted exponential distribution with applications



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ABSTRACT

A two-parameter Inverted Weighted Exponential distribution was derived in this paper. Its various statistical properties were established and the maximum likelihood estimation method was used to estimate the model parameters. Two real life applications were provided to assess the superiority of the Inverted Weighted Exponential distribution over existing distributions.

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1. Introduction

Gupta and Kundu (2009) developed the Weighted Exponential (WE) distribution as a lifetime model and it has been used appreciably both in engineering and medicine (Oguntunde, 2015; Oguntunde et al., 2016). The distribution is versatile and it can compete with other well-known distributions including the Weibull distribution, Gamma distribution and Generalized Exponential distribution.

In Oguntunde et al. (2016), a new version of the WE distribution was studied using the concept of Nasiru (2015) but applications to real life data sets revealed that the existing WE distribution performs better. To this end, the WE distribution due to Gupta and Kundu (2009) is used as the baseline distribution in this present research.

There are several other weighted distributions in the literature. For instance, the Double Weighted Weibull distribution (Saghir and Saleem, 2016), Weighted Inverse Weibull distribution (Sherina and Oluyede, 2014), Modified Double Weighted Exponential Distribution (Saghir et al., 2015) and many more are remarkable examples. These weighted distributions have been applied to real life data sets and they performed better than other competing distributions.

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It is worthy of note that the WE distribution has been generalized using the Exponentiated family of distributions (Oguntunde, 2015) and Transmuted family of distributions (Dar et al., 2017; Yassmen and Abdelall, 2016). Readers can also see Roy and Adnan (2012) for other extensions of the Weighted Exponential distribution. However, a comprehensive review with discussion and characterizations of weighted distributions are available in Saghir et al. (2017).

In this paper however, an inverted version of the WE distribution is introduced and explored. In section 2, the densities of the Inverted Weighted Exponential (IWE) distribution are derived with its various statistical properties, in section 3, an application to two real life data sets are provided.

2. The inverted weighted exponential (IWE) distribution

The densities of the two-parameter WE distribution as proposed by Gupta and Kundu (2009) are:

$$G(x) = 1 - \frac{1}{\alpha} e^{-\beta x} (\alpha + 1 - e^{-\alpha \beta x}); x > 0, \alpha > 0, \beta > 0$$
(1)

and

$$g(x) = \frac{(\alpha+1)}{\alpha} \beta e^{-\beta x} (1 - e^{-\alpha \beta x}); x > 0, \alpha > 0, \beta > 0$$
 (2)

where α represents the shape parameter and β is the scale parameter; G(x) and g(x) are the cdf and pdf of the WE distribution respectively.

From Eq. 1, the cdf of the Inverted Weighted Exponential is derived as follows:

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$$F(x) = 1 - \left[1 - \frac{1}{\alpha}e^{-\frac{\beta}{x}}\left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}}\right)\right]$$

after simple calculations,

$$F(x) = \frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}} \right); x > 0, \alpha > 0, \beta > 0.$$

$$(3)$$

Its corresponding pdf is derived as:

$$f(x) = \frac{(\alpha+1)}{\alpha x^2} \beta e^{-\frac{\beta}{x}} \left(1 - e^{-\frac{\alpha\beta}{x}}\right); x > 0, \alpha > 0, \beta > 0$$

$$\tag{4}$$

where α is the shape parameter and β is the scale parameter

Theorem: Let X denote a continuous random variable, then the pdf of the Inverted Weighted Exponential distribution as derived in Eq. 4 is a valid pdf.

Proof: For the pdf to be valid, then; $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_0^\infty f(x)dx = \frac{(\alpha+1)}{\alpha}\beta \int_0^\infty \frac{e^{-\frac{\beta}{x}}}{x^2} \left(1 - e^{-\frac{\alpha\beta}{x}}\right)dx$$
$$= \frac{(\alpha+1)}{\alpha}\beta \int_0^\infty \frac{e^{-\frac{\beta}{x}}}{x^2}dx - \frac{(\alpha+1)}{\alpha}\beta \int_0^\infty \frac{e^{-\frac{\beta(\alpha+1)}{x}}}{x^2}dx$$

after simple calculations,

$$=\frac{(\alpha+1)}{\alpha}\left[e^{-\frac{\beta}{x}}\right]_{0}^{\infty}-\frac{1}{\alpha}\left[e^{-\frac{\beta(\alpha+1)}{x}}\right]_{0}^{\infty}$$

therefore;

$$\int_0^\infty f(x)dx = 1$$

This completes the proof.

The pdf of the IWE distribution is represented graphically in Fig. 1. Fig. 1 shows that the shape of the Inverted Weighted Exponential distribution could be unimodal (inverted bathtub) or decreasing depending on the parameter values. A possible plot for the cdf of the IWE distribution is presented in Fig. 2.



Fig. 1: PDF of the inverted weighted exponential distribution ($a = \alpha, b = \beta$)



Fig. 2: CDF of the inverted weighted exponential distribution ($\alpha = 0.5, \beta = 0.5$)

2.1. Reliability analysis

Survival function

We express the survival function of the IWE distribution as:

Hazard function

We express the hazard function (failure rate) of the IWE distribution as:

 $S(x) = 1 - \frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}} \right); x > 0, \alpha > 0, \beta > 0.$ (5)

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$$h(x) = \frac{\frac{(\alpha+1)}{\alpha x^2} \beta e^{-\frac{\beta}{\lambda}} \left(1 - e^{-\frac{\alpha\beta}{\lambda}}\right)}{1 - \frac{1}{\alpha} e^{-\frac{\beta}{\lambda}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{\lambda}}\right)}; x > 0, \alpha > 0, \beta > 0.$$
(6)

Possible plots for the failure rate of the IWE distribution are presented in Fig. 3.



Fig. 3: Hazard function of the inverted weighted exponential distribution ($a = \alpha, b = \beta$)

From Fig. 3, it can be seen that the hazard function of the IWE distribution exhibits unimodal (inverted bathtub) and decreasing shapes. This implies that the IWE distribution can be used to describe or model real life phenomena with unimodal or decreasing failure rates.

Odds function

We express the odds function of the IWE distribution as:

$$O(x) = \frac{\frac{1}{\alpha}e^{-\frac{\beta}{x}\left(\alpha+1-e^{-\frac{\alpha\beta}{x}}\right)}}{1-\frac{1}{\alpha}e^{-\frac{\beta}{x}\left(\alpha+1-e^{-\frac{\alpha\beta}{x}}\right)}}; x > 0, \alpha > 0, \beta > 0.$$
(7)

Reversed hazard function

We express the reversed hazard function of the IWE distribution as:

$$r(x) = \frac{\frac{(\alpha+1)}{\alpha x^2} \beta e^{-\frac{\beta}{x}} \left(1 - e^{-\frac{\alpha\beta}{x}}\right)}{\frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}}\right)}; x > 0, \alpha > 0, \beta > 0.$$
(8)

2.2. Order statistics

Let $x_1, x_2, ..., x_n$ denote random samples from a pdf and cdf distributed according to the Inverted Weighted Exponential distribution, the pdf of the *kth* order statistics from the Inverted Weighted Exponential distribution is derived from:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k}.$$
 (9)

Therefore, the pdf of the *kth* order statistics for the Inverted Weighted Exponential distribution is:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \frac{(\alpha+1)}{\alpha x^2} \beta e^{-\frac{\beta}{x}} \left(1 - e^{-\frac{\alpha\beta}{x}}\right) \left[\frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}}\right)\right]^{k-1} \times \left[1 - \frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}}\right)\right]^{n-k}.$$

Therefore, we obtain the distribution of both the minimum and maximum order statistics for the Inverted Weighted Exponential distribution respectively as follows:

$$f_{1:n}(x) = n \frac{(\alpha+1)}{\alpha x^2} \beta e^{-\frac{\beta}{x}} \left(1 - e^{-\frac{\alpha\beta}{x}}\right) \times \left[1 - \frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}}\right)\right]^{n-1}$$
(10)

and

$$f_{n:n}(x) = n \frac{(\alpha+1)}{\alpha x^2} \beta e^{-\frac{\beta}{x}} \left(1 - e^{-\frac{\alpha\beta}{x}}\right) \left[\frac{1}{\alpha} e^{-\frac{\beta}{x}} \left(\alpha + 1 - e^{-\frac{\alpha\beta}{x}}\right)\right]^{n-1}$$
(11)

2.3. Parameter estimation

In this paper, the maximum likelihood estimation (MLE) method is proposed in estimating the two parameters of the Inverted Weighted Exponential distribution. Let $x_1, x_2, ..., x_n$ samples each distributed according to the Inverted Weighted Exponential distribution, then the likelihood function is:

$$f(x_1, x_2, \dots, x_n; \alpha, \beta) = \prod_{i=1}^n \left[\frac{(\alpha+1)}{\alpha x_i^2} \beta e^{-\frac{\beta}{x}} \left(1 - e^{-\frac{\alpha \beta}{x_i}} \right) \right].$$
(12)

Let *L* be the log-likelihood function, then: $L = n \log(\alpha + 1) - n \log(\alpha) - 2 \sum_{i=1}^{n} \log(x_i) + n \log(\beta) - 2 \sum_{i=1}^{n} \log(x_i) + n \log(\beta) = 0$

$$\sum_{i=1}^{n} \frac{\beta}{x_i} + \sum_{i=1}^{n} \log\left(1 - e^{-\frac{\alpha\beta}{x_i}}\right),$$
(13)

solving the resulting non-linear equations of $\frac{dL}{d\alpha} = 0$ and $\frac{dL}{d\beta} = 0$ gives the maximum likelihood estimates of the parameters. The solution cannot be obtained analytically but by numerical methods using appropriate software. For instance, R software was used to obtain the estimates in this research.

3. Application to data sets

For illustration purposes, two real life applications of the IWE distribution are provided. The performance of the IWE distribution was compared with that of the Weighted Exponential distribution using log-likelihood and Akaike Information Criterion (AIC) as selection criteria. The distribution with the lowest negative log-likelihood and AIC values is selected as the 'best' for the data set used.

First data set

This data has been used by Bhaumik et al. (2009) to test the parameters of a Gamma distribution. It

represents the vinyl chloride data (in mg/l) which was obtained from clean upgradient monitoring wells. It consists of thirty-four (34) observations as follows which is summarized in Table 1:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2

Table 2 displays the performance of the competing distributions.

Remark: From Table 2, the IWE distribution has the lowest negative log-likelihood and AIC values, therefore, it can be concluded that it fits the data set better than the Weighted Exponential distribution.

Table 1: Summary of vinyl chloride data set								
n	mean	Q 1	median	Q ₃	variance	skewness	kurtosis	
34	1.879	0.500	1.150	2.475	3.812594	1.603688	5.005408	
Table 2 . Analysis of the performance of the competing distributions I								

Table 2 : Analysis of the performance of the competing distributions I							
Distributions	Estimates	Negative Log-likelihood	AIC				
Inverted Weighted Exponential	$\hat{\alpha} = 0.01218$	101.2751	206.5502				
	$\beta = 5.00093$	10112/01	20000002				
Weighted Exponential	$\hat{\alpha} = 0.009248$	403.9913	811.9826				
	$\beta = 10.00000$	10017710	0110020				

Second data set

This data represents the strengths of 1.5 cm glass fibres. The observations are contained in Oguntunde et al. (2017), Merovci et al. (2016), Bourguignon et

al. (2014), and Smith and Naylor (1987). We provide the summary of the data in Table 3. Table 4 displays the performance of the competing distributions.

	Table 3: Summary of vinyl chloride data set							
_	n	mean	Q_1	media	n Q3	variance	skewness	kurtosis
(63	1.507	1.375	1.590	1.685	0.1050575	-0.8999263	3.923761
			_		_			
Table 4: Analysis of the performance of the competing distributions II								
	Distributions				Estimates	Negativ	Negative Log-likelihood	
Inve	Inverted Weighted Exponential				$\hat{\alpha} = 0.02587$	7	119.8072	
	Weighted Exponential				$\hat{\beta} = 10.0000$	0		
					$\hat{\alpha} = 0.02893$	3	147 1211	
					$\hat{B} = 4.99913$	3	147.1211	

Remark: From Table 4, the IWE distribution has the lowest negative log-likelihood and AIC values, therefore, it can be concluded that it fits the data set better than the Weighted Exponential distribution.

4. Conclusion

The Inverted (Inverse) Weighted Exponential distribution has been derived and studied successfully. The model has unimodal (inverted bathtub) and decreasing shapes (depending on the value of the parameters). Explicit expressions for its basic statistical properties have been successfully derived. The model exhibits unimodal and decreasing failure rates, this implies that the model can be used to describe and model real life phenomena with unimodal or decreasing failure rates. For the real life applications provided, the Weighted distribution Inverted Exponential performs better than the Weighted Exponential

distribution; it is however a good and competitive model as it can be used in place of some other important models in the literature.

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